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Theory of operation of the magnetic crescograph

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In the formulation of the theory of operation of the magnetic crescograph, the concept of rigid body motion of an astatic pair of magnetic needles has been considered It has been shown that under existing conditions, there is only one degree of freedom, involving rotation of the astatic pair about a vertical axis. During the course of analysis it has been shown that the sensitivity of the instrument may be altered by a simple horizontal displacement of the suspending fibre holding the astatic pair, from the tip of the magnetic needle.

Introduction

During his classical researches on photosynthesis, Bose (1924) devised a special radiometer for the determination of the energy of different rays of the solar spectrum by measuring the elongation of a metallic wire coated with lampblack. The particular spectral ray falling on the wire was absorbed and thus raised the temperature proportionately to the energy of radiation. However, the rise of temperature was excessively feeble, being of the order of 10^{-5} °C; the resulting increase of length was so minute as to be undetectable by any method of magnification then available. Bose got over this difficulty by means of a magnetic device called *Magnetic Crescograph*, by which he obtained a magnification of about 50×10^6 times or even higher.

A diagrammatic representation of the apparatus is given in figure 1. W is a length of zinc wire which becomes lengthened by the rise of temperature produced by absorbed radiation. It is attached by a hook to the short arm of a long magnetic lever, the N-end of which is lowered by any elongation of the sensitive wire. In front of the N-end of the magnetic lever is suspended a pair of astatic magnetic needles with an attached mirror. As the N-pole of the magnetic lever is lowered it produces increasing deflection of the suspended astatic pair, which is magnified by the spot of light reflected from the attached mirror.

Using a prototype of the instrument, Chatterjee & Ghosh (1968) measured the magnetostriction effect in ferrite. More recently, Chatterjee & Gupta (1970) have developed a new method for magnifying galvanometer movements, utilizing a modified version of Bose's original magnetic crescograph. A theoretical treatment of the mode of operation of Bose's original instrument is given below.

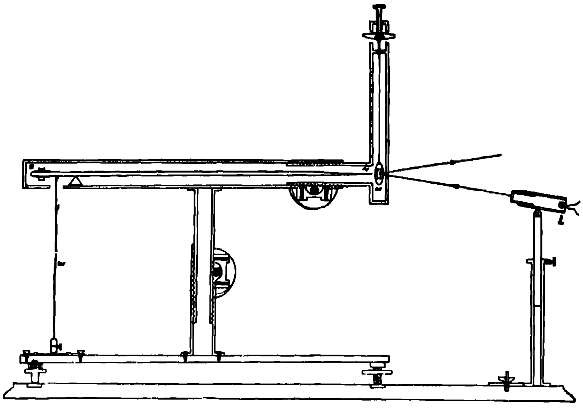

Figure 1

THEORY OF MAGNETIC CRESCOGRAPH

Let the two magnetic needles ns and $n's'$, each of pole strength m, be attached to a vertical rigid lamina (as indicated in figure 2) so that their poles are at the positions:

$$n(+a,\ 0,\ +b), \qquad s\,(+a,\ 0,\ -b),$$
$$n'(-a,\ 0,\ -b), \qquad s'(-a,\ 0,\ +b),$$

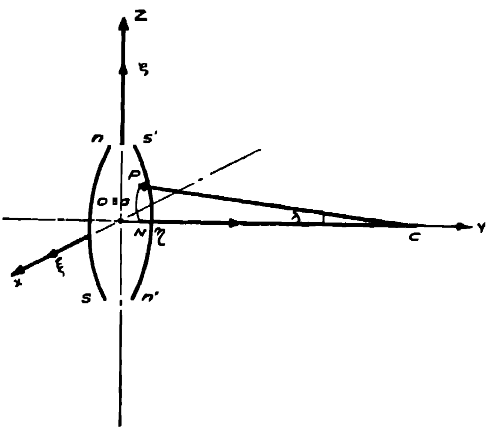

Figure 2

with reference to a body axes frame $G\xi\eta\zeta$ which is along the principal axes system of the lamina and with its origin at the centre of gravity G. These positions of the poles with reference to $G\xi\eta\zeta$ are the same for all positions and orientations of the lamina. A fixed frame of reference $Oxyz$ is so selected that initially $Oxyz = G\xi\eta\zeta$ and Oz-is vertical. Let an isolated N-Pole of strength M with its initial position with respect to $Oxyz$ system at $N(0, +f, 0)$, be allowed to move in the circle :

$$(y-f-R)^2+z^2 = R^2$$

about the point $C(0, f+R, 0)$, such that at any instant of time t, its position with respect to fixed frame $Oxyz$ is $P[0, f+R(1-\cos\lambda), R\sin\lambda]$. Due to magnetic attraction and repulsion the vertical lamina containing the magnetic needles will suffer a displacement which is expected to be both translational and rotational

Due to translational motion the centre of gravity G will be shifted to a new position $G(x_0, y_0, z_0)$, as referred to fixed $Oxyz$, while rotational displacements are determined by Eulerian angles ϕ, θ, ψ If (ξ, η, ζ) be the position of a point with respect to moving frame $G\xi\eta\zeta$ and if (x, y, z) be the position of the same point with respect to fixed frame $Oxyz$, then the rule of transformation from $G\xi\eta\zeta$ system to $Oxyz$ system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} \cos\psi\cos\phi & \cos\psi\sin\phi & \sin\psi\sin\theta \\ -\cos\theta\sin\phi\sin\psi, & +\cos\theta\cos\phi\sin\psi, & \\ -\sin\psi\cos\phi & -\sin\psi\sin\phi & \cos\psi\sin\theta \\ -\cos\theta\sin\phi\cos\psi, & +\cos\theta\cos\phi\cos\psi, & \\ \sin\theta\sin\phi & -\sin\theta\cos\phi, & \cos\theta \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad \ldots (1)$$

By the rule of transformation (1), the position $n(a, 0, b)$ wth respect to moving frame $G\xi\eta\zeta$ becomes

$$n[x_0+a(\cos\psi\cos\phi-\cos\theta\sin\phi\sin\psi)+b\sin\psi\sin\theta,$$
$$y_0-a(\sin\psi\cos\phi+\cos\theta\sin\phi\cos\psi)+b\cos\psi\sin\theta,$$
$$z_0+a\sin\theta\sin\phi+b\cos\theta]$$

referred to fixed $Oxyz$ frame, and similar co-ordinates referred to fixed frame $Oxyz$ for s, n', s'. Since the instantaneous position of the N-Pole is $P[0, f+R(1-\cos\lambda)$, $R\sin\lambda]$ referred to fixed frame $Oxyz$, the distance

$$P_n = [\{x_0+a(\cos\psi\cos\phi-\cos\theta\sin\phi\sin\psi)+b\sin\psi\sin\theta\}^2$$
$$+\{(y_0-R.\overline{1-\cos\lambda}-f)-a(\sin\psi\cos\phi+\cos\theta\sin\phi\cos\psi)+b\cos\psi\sin\theta\}^2$$
$$+\{(z_0-R.\sin\lambda)+a\sin\theta\sin\phi+b\cos\theta\}^2]^{\frac{1}{2}}$$

and similar expressions for Pn', Ps, Ps'.

Since the magnetic needles form an astatic pair, the geomagnetic field will have no effect on the lamina. The only controlling forces are the torsion generated in the fibre suspending the lamina and the weight W of the lamina which acts at G vertically downwards The deflecting forces are the forces of magnetic attraction and repulsion between the isolated N-Pole at P and each of the poles n, s, n', s'.

To calculate the kinetic and potential energy of the system, the different angular velocities about the principal axes are

(i) $(\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi)$ about $G\xi$

(ii) $(\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi)$ about $G\eta$

and (iii) $(\dot{\phi} \cos\theta + \dot{\psi})$ about $G\zeta$,

where dot represents the derivative with respect to time t. If χ be the rotation about $G\zeta$, then

$$\chi = \int_0^t (\dot{\phi} \cos\theta + \dot{\psi}) dt = \psi + \phi \cos\theta + \int_0^t (\phi \sin\theta).\dot{\theta}.dt.$$

If I_ξ, I_η, I_ζ be the principal moments of inertia of the moving lamina, then the instantaneous kinetic energy is

$$T = \tfrac{1}{2} \Big[\frac{W}{g} (\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) + I_\xi (\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi)^2$$

$$+ I_\eta (\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi)^2 + I_\zeta (\dot{\phi} \cos\theta + \dot{\psi})^2 \Big]$$

If the suspending fibre be attached to the lamina at $L(0, 0, p)$ referred to moving frame $G\xi\eta\zeta$, then the twist of the fibre is χ and the instantaneous potential energy arising due to rotation χ is $\frac{1}{2} c\chi^2$ where c is the couple per unit angle of twist of the suspending fibre

If V_0 be the potential energy of the system when it is at rest, then initially the instantaneous potential energy is

$$V = V_0 + Wz_0 + \frac{Mm}{Pn} + \frac{Mm}{Pn'} - \frac{Mm}{Ps} - \frac{Mm}{Ps'} + \frac{1}{2} c\chi^2.$$

Hence the Lagrangian function L is

$$L = T - V$$

$$= \tfrac{1}{2} \Big[\frac{W}{g} (\dot{x}_0^2 + \dot{y}_0^2 + \dot{x}_0^2) + I_\xi (\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi)^2$$

$$+I_{\eta}(\dot{\phi}\sin\theta\cos\psi-\dot{\theta}\sin\psi)^2+I_{\zeta}(\dot{\phi}\cos\theta+\dot{\psi})^2\Big]$$

$$-\Big[V_0+Wz_0+\frac{Mm}{Pn}+\frac{Mm}{Pn'}-\frac{Mm}{Ps}-\frac{Mm}{Ps'}+\tfrac{1}{2}c.\chi^2\Big].$$

The six-co-ordinates, $x_0, y_0, z_0, \theta, \phi, \psi$ completely describe the motion of the lamina under initial conditions

$$\dot{x}_0=\dot{y}_0=\dot{z}_0=0 \quad \text{and} \quad \dot{\phi}=\dot{\theta}=\dot{\psi}=0 \qquad \text{at} \quad t=0,$$

and $$x_0=y_0=z_0=0 \quad \text{and} \quad \phi=\theta=\psi=0 \qquad \text{at} \quad t=0.$$

Lagrange's equations for the co-ordinates x_0, y_0, z_0 are

$$\ddot{x}_0+g.\frac{Mm}{W}\cdot\frac{\partial}{\partial x_0}\Big[\frac{1}{Pn}+\frac{1}{Pn'}-\frac{1}{Ps}-\frac{1}{Ps'}\Big]=0 \qquad \ldots \text{(i)}$$

$$\ddot{y}_0+g.\frac{Mm}{W}\cdot\frac{\partial}{\partial y_0}\Big[\frac{1}{Pn}+\frac{1}{Pn'}-\frac{1}{Ps}-\frac{1}{Ps'}\Big]=0 \qquad \ldots \text{(ii)}$$

$$\ddot{z}_0+g.\frac{Mm}{W}\cdot\frac{\partial}{\partial z_0}\Big[\frac{1}{Pn}+\frac{1}{Pn'}-\frac{1}{Ps}-\frac{1}{Ps'}\Big]+g=0 \qquad \ldots \text{(iii)}$$

Integrating the equation (i) with respect to x_0, the equation (ii) with respect to y_0 and equation (iii) with respect to z_0, respectively,

$$\tfrac{1}{2}\dot{x}_0^2=-g\frac{Mm}{W}[Pn^{-1}+Pn'^{-1}-Ps^{-1}-Ps'^{-1}]$$

$$\tfrac{1}{2}\dot{y}_0^2=-g\frac{Mm}{W}[Pn^{-1}+Pn'^{-1}-Ps^{-1}-Ps'^{-1}]$$

$$\tfrac{1}{2}\dot{z}_0^2=-g.\frac{Mm}{W}[Pn^{-1}+Pn'^{-1}-Ps^{-1}-Ps'^{-1}]-gz_0,$$

the constants of integration vanishing under initial conditions.

One Degree Freedom for Small Displacements

It may be seen that so far as x_0, y_0, z_0 are small in order to satisfy the conditions

$$|x_0| < a+b-\{(a+b)^2-\tfrac{1}{8}(a^2+b^2)\}^{\frac{1}{2}},$$

$$|y_0| < a+b-\{(a+b)^2-\tfrac{1}{8}(a^2+b^2)\}^{\frac{1}{2}}+R.(1-\cos\lambda)+f,$$

$$|z_0| < a+b-\{(a+b)^2-\tfrac{1}{8}(a^2+b^2)\}^{\frac{1}{2}}+R\sin\lambda,$$

$$Pn^{-1} = F^{-\frac{1}{2}} \cdot \Big[1-\frac{1}{F}\{ax_0(\cos\psi\cos\phi-\cos\theta\sin\phi\sin\psi)+bx_0\sin\psi\sin\theta$$

$$-a(y_0-R.\overline{1-\cos\lambda}-f)\,.\,(\sin\psi\cos\phi+\cos\theta\sin\phi\cos\psi)$$

$$+b(y_0-R.\overline{1-\cos\lambda}-f)\,.\,\cos\psi\sin\theta+a(z_0-R.\sin\lambda)\,.\,\sin\theta\sin\phi$$

$$+b(z_0-R.\sin\lambda).\,\cos\theta\}\Big], \quad \text{approx.}$$

and similar approximations for Pn'^{-1}, Ps^{-1}, Ps'^{-1},
where

$$F = x_0^2+(y_0-R.\overline{1-\cos\lambda}-f)^2+(z_0-R\sin\lambda)^2+a^2+b^2.$$

These approximate values show that the expression $Pn^{-1}+Pn'^{-1}-Ps^{-1}-Ps'^{-1}$ vanishes upto the first order. Hence

$$\tfrac{1}{2}\dot{x}_0^2 = 0, \quad \tfrac{1}{2}\dot{y}_0^2 = 0, \quad \tfrac{1}{2}\dot{z}_0^2 = -gz_0 \quad \text{for all times.}$$

First two equations involving $\dot{x}_0$ and $\dot{y}_0$, under initial conditions, give

$$x_0 = y_0 = 0 \quad \text{for all times,}$$

while the last one involving $\dot{z}_0$ and z_0 implies that only non-positive values of z_0 are possible. If z_0 be negative, G will be lowered which means an elongation of the suspending fibre. If the fibre having an unchanged length with its upper end at $O\,(0, 0, d)$ referred to fixed frame $Oxyz$, and lower end at $L(0, 0, p)$ in $G\xi\eta\zeta$-frame or by (1), at $L(x_0+p\sin\psi\sin\theta,\ y_0+p\cos\psi\sin\theta,\ z_0+p\cos\theta)$ in $Oxyz$ frame then its present length equals to its initial length,—a condition yielding

$$(x_0+p\sin\psi\sin\theta)^2+(y_0+p\cos\psi\sin\theta)^2+(d-z_0-p\cos\theta)^2 = (d-p)^2 \qquad (2)$$

Since $x_0 = y_0 = 0$ this equation (2) reduces to

$$p^2\sin^2\theta+(d-z_0-p\cos\theta)^2 = (d-p)^2$$

For a possible negative value z_0 may be substituted by $-\ |z_0|$ and as a result

$$\cos\theta = \frac{2|z_0|.d+2p\,d+|z_0|^2}{2.p.d+2|z_0|.p} \leqslant 1 \text{ since } -\pi/2 < \theta < \pi/2.$$

Hence $$|z_0|\,[\,|z_0|+2(d-p)] \leqslant 0$$

As both $|z_0|$ and $d-p$ are positive, the only possibility is

$$|z_0| = 0$$

Therefore,

$$x_0 = y_0 = z_0 = 0 \quad \text{for all times.}$$

In view of this result the relation (2) produces $\theta = 0$ which further proves

$$\chi = \phi+\psi; \quad \dot{\chi} = \dot{\phi}+\dot{\psi},$$

$$Pn = [a^2+\{f+R.\overline{1-\cos\lambda}\}^2+(b-R\sin\lambda)^2$$
$$+2a\{f+R.\overline{1-\cos\lambda}\}.\sin\chi]^{\frac{1}{2}}$$

and similarly for Ps, Pn', Ps'.

In this case, the Lagrangian function L becomes

$$L = \tfrac{1}{2} I_\zeta.\dot{\chi}^2-\left[V_0+\frac{Mm}{Pn}+\frac{Mm}{Pn'}-\frac{Mm}{Ps}-\frac{Mm}{Ps'}+\tfrac{1}{2}c.\chi^2\right]$$

Langrange's equation for the co-ordinate χ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\chi}}\right)-\frac{\partial L}{\partial\chi}=0$$

which gives

$$\ddot{\chi}+\beta^2\dot{\chi}=\frac{Mm}{I_\zeta}[Pn^{-3}+Ps'^{-3}-Pn'^{-3}-Ps^{-3}]$$
$$\times a\{f+R(1-\cos\lambda)\}.\cos\chi$$

where $\beta^2=\dfrac{c}{I_\zeta}=\dfrac{4\pi^2}{\tau^2}$ [τ = free period of oscillation of the lamina.]

Since the lamina moves in air, there must be a damping term on the left side of the equation and so

$$\ddot{\chi}+2\alpha.\dot{\chi}+\beta^2\chi=\frac{Mm}{I_\zeta}[Pn^{-3}-P_{n'}^{-3}+Ps'^{-3}-Ps^{-3}]$$
$$\times a\{f+R.(1-\cos\lambda)\}\cos\chi \quad \cdots \quad (3)$$

2α being the co-efficient of air damping.

For small values of χ and so for small values of λ, $\cos\chi$ and $\cos\lambda$, either of them may be replaced by unity while $\sin\chi$ and $\sin\lambda$ by χ and λ, respectively.

In this case

$$P_n=[a^2+f^2+(b-R\lambda)^2+2af\chi]^{\frac{1}{2}}$$
$$Ps=[a^2+f^2+(b+R\lambda)^2+2af\chi]^{\frac{1}{2}}$$
$$Pn'=[a^2+f^2+(b+R\lambda)^2-2af\chi]^{\frac{1}{2}}$$
$$Ps'=[a^2+f^2+(b-R\lambda)^2-2af\chi]^{\frac{1}{2}}.$$

If λ be such as to satisfy

$$|R\lambda| < (a^2+2b^2+f^2)^{\frac{1}{2}}-b,$$

the approximate value of Pn^{-3} is

$$Pn^{-3} \doteq [a^2+b^2+f^2]^{-3/2}.\left[1+\frac{3(bR\lambda-\frac{1}{2}R^2\lambda^2-af\chi)}{a^2+b^2+f^2}\right]$$

and similarly for Ps^{-3}, Pn'^{-3} and Ps'^{-3}.

These approximate values indicate that upto the first order of approximation

$$Pn^{-3}+Ps'^{-3}-Pn'^{-3}-Ps^{-3} = \frac{12bR\lambda}{[a^2+b^2+f^2]^{5/2}}$$

Hence the equation of motion (3) for the deflection χ is,

$$\ddot{\chi}+2\alpha.\dot{\chi}+\beta^2\chi = \frac{12Mm}{I_\zeta}\cdot\frac{abfR}{[a^2+b^2+f^2]^{5/2}}\cdot\lambda \qquad \ldots \quad (4)$$

provided $\quad |R\lambda| < (a^2+2b^2+f^2)^{\frac{1}{2}}-b$

This differential equation may be solved and χ may be obtained as a function of time t, provided the actual form of the variable λ as a function of time t is known.

Sensitivity

Let the final steady value of λ be λ_f and that of χ be X_f. Then for final steady state, the equation (4) takes the form

$$\beta^2 X_f = \frac{12Mm}{I_\zeta}\cdot\frac{abfR}{(a^2+b^2+f^2)^{5/2}}\cdot\lambda_f \qquad \ldots \quad (4a)$$

which gives

$$S = \frac{X_f}{\lambda_f} = \frac{12Mm}{c}\cdot\frac{abfR}{(a^2+b^2+f^2)^{5/2}} \qquad \ldots \quad (5)$$

a quantity which is a measure of the amplifying capacity of the instrument and may be termed as *sensitivity* of the instrument. The expression (5) shows that S increases with the increase of either of the quantities Mm, R and $1/c$. Also S depends upon the factor

$$\frac{f}{(a^2+b^2+f^2)^{5/2}}$$

For given values of a, b, c Mm and R, S attains its maximum value

$$S_{max} = \frac{16\times 12}{25.\sqrt{5}}\cdot\frac{Mm}{c}\cdot\frac{abR}{(a^2+b^2)^2} \qquad \ldots \quad (6)$$

when $f = \frac{1}{2}(a^2+b^2)^{\frac{1}{2}}$.

In view of the relation (4a), the equation (4) may be written as

$$\ddot{\chi}+2\alpha\,.\,\dot{\chi}+\beta^2\chi = \beta^2\,.\,X_f\,.\,\frac{\lambda}{\lambda_f} \qquad \ldots \quad (7)$$

This equation (7) completely describes the motion of the suspended part of the crescograph for small values of χ and λ. The equation (5) determines amplification for a particular setting of the instrument. If there be an arrangement for varying the quantity f, the distance of the centre of gravity G of the suspended part from the deflecting pole N, the maximum sensitivity (6) of the instrument may be reached when the particular value $f = \frac{1}{2}(a^2+b^2)^{\frac{1}{2}}$ is adjusted.

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